Arbitrary Splitting of Taxicab Angles

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Abstract

A construction to split a taxicab angle into an arbitrary number of equal angles is presented.

In Euclidean geometry, an arbitrary angle can be bisected using only a compass and an unmarked straight edge. But, angle trisection is not possible as was proven by Pierre Wantzel in 1837 [1].

In taxicab geometry, the t-radian [2] is the direct analogue of the radian in Euclidean geometry. As noted by Dawson [3] and others, the t-radian can be split into an arbitrary number of equal angles. This is a direct result of the t-radian being defined as the length of a line segment and the fact that a line segment can be split into an arbitrary number of equal parts. With taxicab geometry routinely failing to satisfy theorems taken for granted in Euclidean geometry, such a result is a refreshing change of pace!

While Dawson is correct in his statement, and he provides a construction to bisect a t-radian angle, it was not immediately obvious whether the arbitrary splitting of an angle could be done with only taxicab constructs. A line segment can be arbitrarily split in Euclidean geometry, so Euclidean constructs could be used to arbitrarily split a t-radian angle. But, what would this construction look like with only taxicab constructs? The goal of this paper is simply to extend Dawson's construction.

CONSTRUCTION 1 *To split a line segment with slope 1 into $n \geq 3$ equal parts using taxicab constructions*

Given a line segment $\overline{AB}$ of length $2l$ with slope 1 with A at the origin, construct a taxicab circle of radius $2l$ with center at B (Figure 1). Likewise, construct a taxicab circle of radius $2l$ with center at A. In $n > 3$, at the intersection of the circle around A with the extension of the line segment, construct a circle of radius $2l$. If $n > 4$, repeat the last step $n-4$ additional times. Construct a line from the bottom corner of the last circle to point B creating an intersection P with the circle about B. Construct a line from P to the top of the circle about A creating an intersection C with the line segment $\overline{AB}$. The taxicab length of line segment $\overline{AC}$ is $2l/n$.

PROOF: Let $A = (0, 0)$ and $B = (l, l)$. The bottom corner of the last constructed circle is located at $((3-n)l, (1-n)l)$ so the line from such a point to B is

$$y = \frac{n}{n-2}(x - l) + l$$

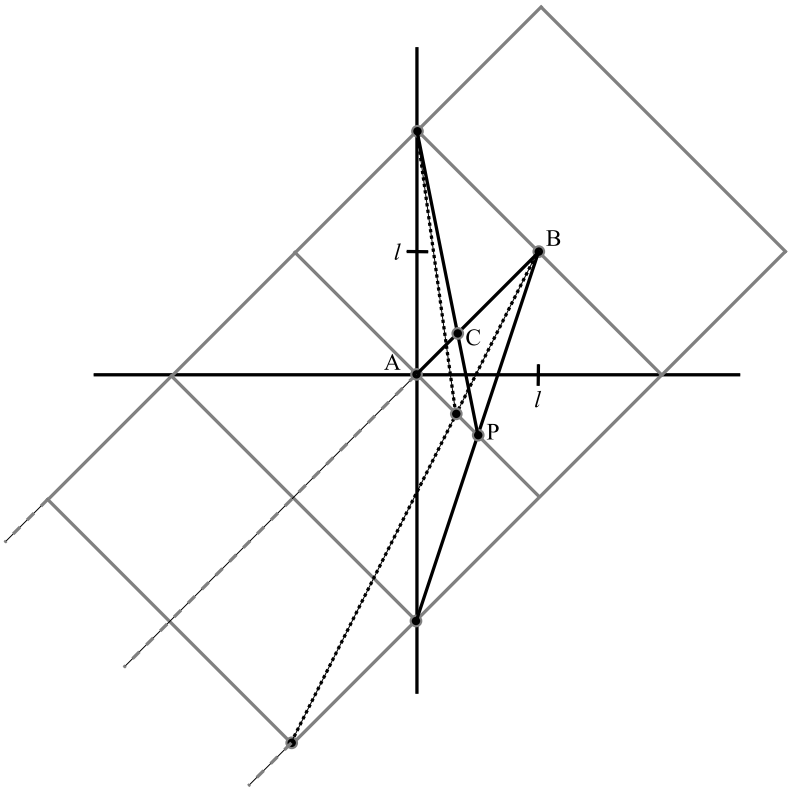

Figure 1: Construction to split a line segment into n equal parts ($n = 3$ and $n = 4$ shown).

This line intersects the line $y = -x$ at point P with x-coordinate $x = \frac{l}{(n-1)}$. The line from this intersection to the point $(0, 2l)$ at the top of the circle centered at A is

$$y = (1 - 2n)x + 2l$$

This line intersects the line $y = x$ at $x = \frac{l}{n}$ giving a taxicab distance from A to this point C of

$$d(A, C) = \left|\frac{l}{n} - 0\right| + \left|\frac{l}{n} - 0\right| = \frac{2l}{n}$$

□

Dawson's construction for bisection is a special case of the above construction with an adjustment because the bottom corner of the last constructed circle is not defined for $n = 2$. This step is merely skipped and the bottom corner of the circle about A is connected to the top of the circle about B which completes the construction.

The presented construction need not be limited to lines of slope 1. This was merely done for the simplicity of calculation and because t-radian angles

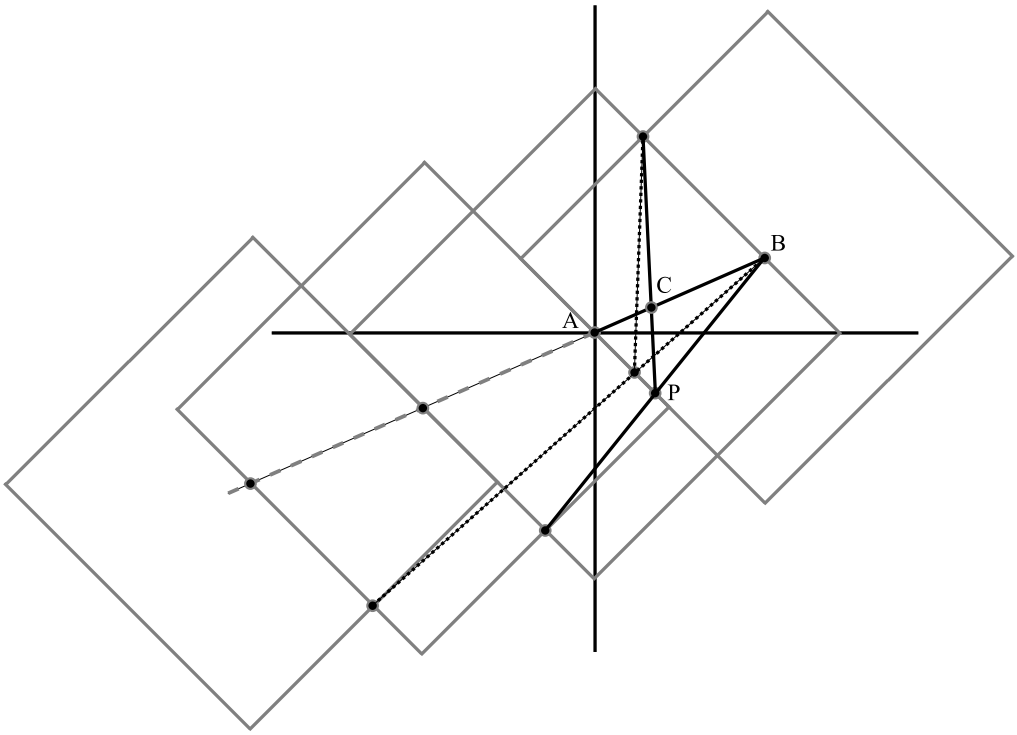

Figure 2: Generalized construction to split a line segment into n equal parts ($n = 3$ and $n = 4$ shown).

are defined along diagonal lines. Figure 2 illustrates the general construction. Not only have we developed a method to arbitrarily split taxicab angles and line segments, but we have also developed yet another method for arbitrarily splitting Euclidean line segments!

Many times there are multiple constructions that accomplish the same result. The challenge now is to find other taxicab constructions to arbitrarily split a line segment into equal parts!

References

[1] Wantzel, P. L. *Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas*, Journal de Mathématiques Pures et Appliquées, Vol. 1, No. 2 (1837), pp. 366-372.

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[3] Dawson, Robert J. MacG. *Crackpot Angle Bisectors!* Mathematics Magazine, Vol. 80, No. 1 (Feb 2007), pp. 59-64.